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Tutorial

Cylindrical Algebraic Decomposition

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Preface

Goal: To compute symbolically with semi-algebraic sets in practice.

Definition: The *semi-algebraic sets* in $\mathbb{R}^n$ are defined recursively by:

1. the set of points satisfying $p \; \sigma \; 0$, where $p \in \mathbb{R}[x_1, \ldots, x_n]$ and $\sigma \in \{=, \neq, <, \leq, >, \geq\}$, is semi-algebraic

2. the complement of a semi-algebraic set, and the union or intersection of finitely many semi-algebraic sets are semi-algebraic.

Preface

Semi-algebraic sets are usually represented by *Tarski Formulas*.

> **Definition:** A *Tarski Formula* is boolean combination of polynomial equalities and inequalities.

Tarski formulas map directly to the definition of semi-algebraic sets:

$$x_1 \neq 0 \ \wedge \ x_2^2 - 4x_1x_3 > 0 \qquad \longrightarrow \qquad x_1 \neq 0 \ \cap \ x_2^2 - 4x_1x_3 > 0$$

$$x_1^2 + x_2^2 - 1 \leq 0 \Longrightarrow x_1 + x_2 \geq 0 \quad \longrightarrow \quad \overline{x_1^2 + x_2^2 - 1 \leq 0} \ \cup \ x_1 + x_2 \geq 0$$

So we may view semi-algebraic sets from the standpoint of *geometry* or of *logic*. One viewpoint: the *objects* are geometric, the *representations* are logical.

Introduction

What is “Cylindrical Algebraic Decomposition”?

Cylindrical Algebraic Decomposition (CAD) provides an *explicit* representation for semi-algebraic sets, in which many operations and queries can be carried out easily.

Tutorial Outline

- *Motivate*
- *Develop intuition*
- *Present CAD fundamentals more rigorously*
- *Present what you need to use and adapt CAD effectively*
- *Analyze some case studies*

Problems

1. An application from epidemiology

2. Triangles and Euclidean geometry

3. The role of Rolle's theorem

An Epidemiological Problem

- Andreas Weber and colleagues have applied symbolic tools to epidemiological models.
- We consider the SEIT model, used to model tuberculosis in *van den Dreissche & Watmough, 2002*.
- Uses system of ODEs to model movement of disease through population. Model contains many parameters.
- Question: For what parameters is there an *endemic equalibirum*, i.e. an equilibrium that doesn't have the disease dying out.

An Epidemiological Model

$$
\begin{aligned}
S' &= d - dS - \beta_1 IS \\
E' &= \beta_1 IS + \beta_2 IT - (d + \nu + r_1)E + (1 - q)r_2 I \\
I' &= \nu E - (d + r_2)I \\
T' &= -dT + r_1 E + qr_2 I - \beta_2 TI
\end{aligned}
$$

S susceptibles
E exposed (not yet infectious)
I infectious
T under treatment

β_1, β_2 transmission parameters for S and T
d birth and death rate (assumed equal)
ν rate of change from exposed to infectious
r_1, r_2 treatment rates for E and I
q fraction of infectious successfully treated

Endemic equilibrium

- an endemic equilibrium satisfies $0 = S', E', I', T'$ and $0 < S, E, I, T$
- for what parameter values is there a solution satisfying $0 < S, E, I, T$ for:

$$
\begin{aligned}
0 &= d - dS - \beta_1 IS \\
0 &= \beta_1 IS + \beta_2 IT - (d + \nu + r_1)E + (1 - q)r_2 I \\
0 &= \nu E - (d + r_2)I \\
0 &= -dT + r_1 E + qr_2 I - \beta_2 TI
\end{aligned}
$$

assuming all parameters positive?

Endemic equilibrium problem

- It's easy to solve for E,I and T in terms of S three of the equations. Substituting the results into the fourth gives $P = 0$, where

$$
\begin{aligned}
P = & -\nu S^2\beta_1^2 + \beta_1\nu S^2\beta_2 + d\beta_1 S r_2 - d^2\beta_2 S + d^2\beta_1 S + \beta_1 S r_1 r_2 - d\nu S\beta_2 \\
& + \nu\beta_1 S q r_2 - d\beta_2 r_2 S + d\nu S\beta_1 - \beta_1 S\nu\beta_2 + \beta_1 S r_1 d + \beta_2 d^2 + \nu\beta_2 d + \beta_2 d r_2
\end{aligned}
$$

- The condition $0 < S, E, I, T$ is easily seen to be equivalent to $0 < S < 1$.

- So there is an endemic equilibrium for any assignment of positive parameter values for which there is a a real value S such that $P(S) = 0 \;\wedge\; 0 < S < 1$, i.e.

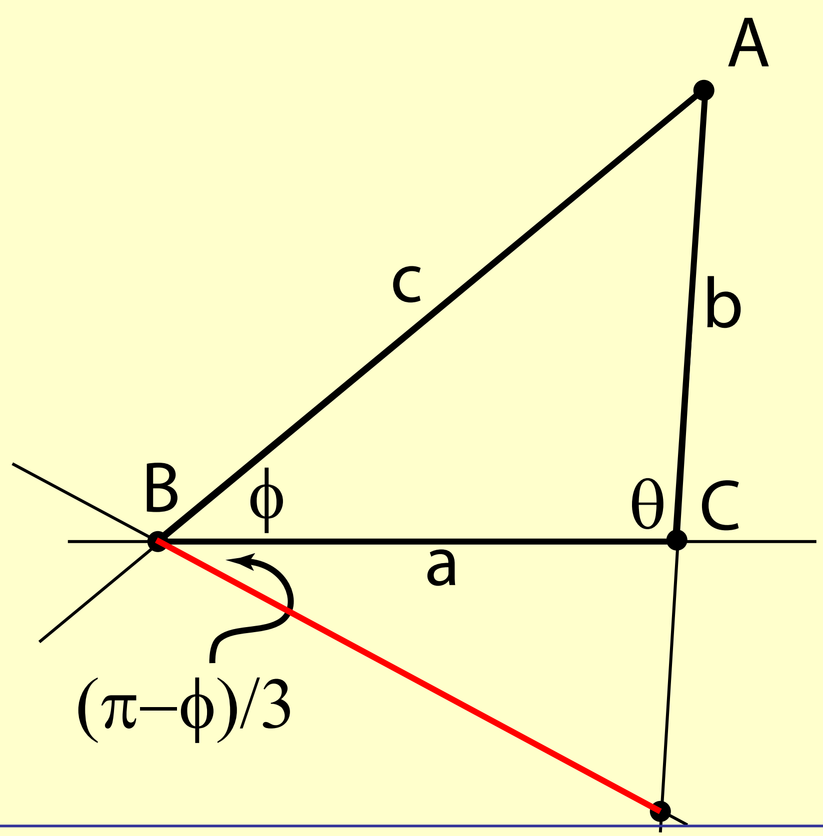
$$\exists S\,[P(S) = 0 \;\wedge\; 0 < S < 1]$$

External Trisector

Given triangle ABC, consider the "external trisector of B with respect to A" as defined in the figure.

Clearly, the trisectors exist if and only if $(\pi - \phi)/3 < \theta$.

Problem: Characterize the existence of the external trisector of B with respect to A in terms of the side lengths a, b, c.



Converting from angles to side lengths

Using standard trig identities we see that $(\pi - \phi)/3 < \theta$ is equivalent to

$$\cos\theta \le \cos\frac{\pi}{3} \ \vee \ \cos\theta > \cos\frac{\pi}{3} \wedge \cos\phi < -4\cos^3\theta + 3\cos\theta$$

Using the law of cosines to describe $\cos\theta$ and $\cos\phi$ in terms of the side lengths a, b and c, and clearing denominators, we get

$$a^2+b^2-c^2 < ab \vee a^2+b^2-c^2 \ge ab \wedge -c\left(a^2+b^2-c^2\right)^3 + 3a^2b^2c\left(a^2+b^2-c^2\right)$$

The External Trisector Problem

In the triangle with side lengths a, b, c, the external trisector of B w.r.t. A exists if and only if:

$$a^2+b^2-c^2 < ab \vee a^2+b^2-c^2 \geq ab \wedge -c\left(a^2+b^2-c^2\right)^3+3a^2b^2c\left(a^2+b^2-c^2\right)$$

Problem: Is there a simpler characterization?

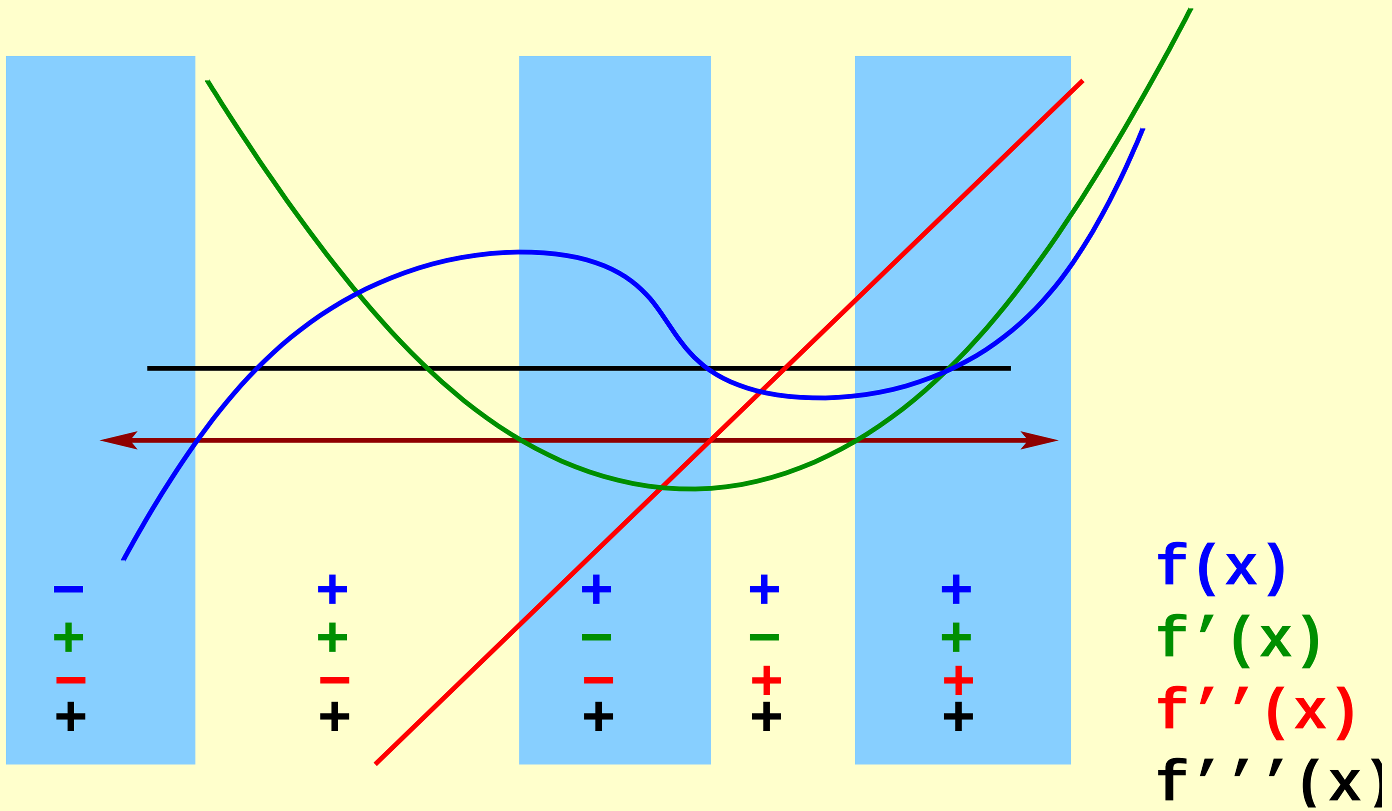
Rolle's Theorem

- Background: from Bruce Anderson's PhD thesis, with Moss Sweedler, Cornell, 1992.

- Rolle's theorem: between any two zeros of $f(x)$ there is a zero of $f'(x)$.

- Generically: from $x = -\infty$ to $x = +\infty$ the signs of f and f' can be $\boxed{\begin{array}{ll} f & +--+ \\ f' & --++ \ \ldots \end{array}}$, but cannot be $\boxed{\begin{array}{ll} f & +-+ \\ f' & --- \ \ldots \end{array}}$ according to Rolle's theorem.

Sequence of "Sign-Stacks"



Sign-stack sequence restrictions

Consider the sign-stack sequence of a generic, monic, univariate polynomial.

1. *monic* implies: the bottom entry is always $+$, the first sign-stack is alternating $+$'s and $-$'s, and the last sign-stack is all $+$'s

2. *generic* implies: consecutive sign-stacks differ in only one entry

3. *Rolle's theorem* implies: from one sign-sequence to the next, an entry may only be changed to equal the entry below it

A sequence satisfying these requirements is called *legal*.

“Rolle’s Theorem Problem”

Problem: is every legal sign-stack sequence realizable by a polynomial?

There are lots of legal sign-stack sequences!

degree of f	1	2	3	4	5	6	7
# legal sequences	1	2	6	42	1000	114650	77740200

Questions?