NAME: **ALPHA:**
INSTRUCTOR: **SECTION:**

CALCULUS I FINAL EXAM SM121, SM121A, SM131
0755-1055 Saturday 10 December 2005

CALCULATORS ALLOWED—SHOW ALL WORK ON THIS TEST

PART ONE: Multiple choice (50%). The first 20 problems are multiple-choice. Fill in the best answer on the bubble sheet. **Write your name, alpha code, and section on your bubble sheet and bubble in your alpha code.** There is no penalty for wrong answers on multiple-choice. Show all your scratch work on this test.

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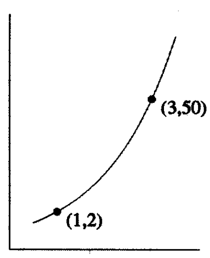
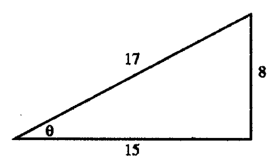
1. The curve on the left below shows part of the graph of a function $f(x) = Ca^x$. The value of a is:
a) −5 b) 2 c) 3 d) 5 e) 50

$$\begin{cases} Ca^3 = 50 \\ Ca^1 = 2 \end{cases} \qquad \frac{Ca^3}{Ca^1} = a^2 = \frac{50}{2} = 25$$
$$a = 5$$





2. For the triangle shown above right, the value of $\csc\theta$ equals:
a) 17/8 b) 8/17 c) 17/5 d) 5/17 e) 15/8

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3. If $\ln A = 6$ and $\ln B = 2$, then $\ln(\sqrt{A}/B^3)$ equals:
a) 9 b) 3 c) 0 d) −3 e) None of these

$$\ln(\sqrt{A}/B^3) = \frac{1}{2}\ln A - 3\ln B = \frac{6}{2} - 3\cdot 2 = -3$$

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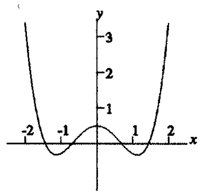
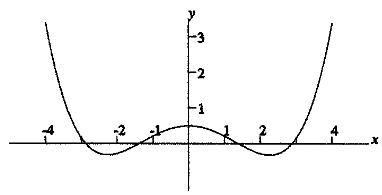
4. The graph below left shows $y = f(x)$ and that below right shows $y = g(x)$. The function $g(x)$ equals:

a) $f(2x)$ b) $f(x)/2$ c) $2f(x)$ d) $f(x-2)$ e) $f(x/2)$





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5. Let

$$f(x) = \begin{cases} -x, & \text{if } x < 0; \\ x^2, & \text{if } 0 \le x < 1; \\ x - 1, & \text{if } 1 \le x < 2; \\ 1, & \text{if } 2 \le x. \end{cases}$$

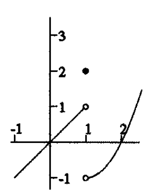
$-0 = 0^2$

$1^2 \neq 1-1$

$2-1 = 1$

Then f is continuous at the x-values:

a) 0 only b) 1 only c) 2 only d) 0 and 1 e) 0 and 2



6. Suppose the graph of f is as shown in the diagram above. Then

$$\lim_{x \to 1+} f(x) =$$

a) -1 b) 0 c) 1 d) 2 e) Does not exist

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The table below is to be used for problems 7 and 8. It shows the position $x(t)$ (measured in miles) of a car at a time t (measured in minutes).

t	0	10	20	30	40	50
$x(t)$	12	22	30	37	42	46

7. The average speed of the car (in miles per hour) between $t = 10$ and $t = 40$ is:
a) 20 b) 30 c) 37 d) 40 e) None of these

$$\frac{42-22}{40-10} = \frac{20 \text{ mi}}{30 \text{ min}} = 40 \text{ mph}$$

8. A reasonable estimate for the instantaneous speed (in miles per hour) of the car at time $t = 50$ is:
a) 4 b) 10 c) 24 d) 42 e) 46

$$\frac{46-42}{50-40} = \frac{4 \text{ mi}}{10 \text{ min}} = 24 \text{ mph}$$

The table below is to be used for problems 9, 10, and 11. Assume that the functions f and g are differentiable on the interval [0,10].

x	1	2	3	4
$f(x)$	−2	0	2	3
$f'(x)$	3	2	−2	4
$g(x)$	−1	5	4	−3
$g'(x)$	4	7	−3	2

9. $(f \circ g)(3) =$ $f(g(3)) = f(4) = 3$
a) 1 b) 2 c) 3 d) 4 e) 5

10. $(f \circ g)'(3) =$ $f'(g(3))\,g'(3) = f'(4)(-3) = 4(-3) = -12$
a) −12 b) 0 c) 6 d) 12 e) Undefined

11. $(fg)'(3) =$ $f(3)\,g'(3) + f'(3)\,g(3) = (2)(-3) + (-2)(4) = -14$
a) −16 b) −14 c) −6 d) 6 e) Undefined

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12. The slope of the line tangent to the curve with equation $y^5 + 4xy - x^3 = 1$ at the point $(2, 1)$ is:
a) −13/8 b) −8/13 c) 8/13 d) 13/8 e) None of these

$$5y^4\,dy + 4x\,dy + 4y\,dx - 3x^2dx = 0$$
$$5y^4\,dy + 4x\,dy = 3x^2dx - 4y\,dx$$
$$5\cdot1^4\,dy + 4\cdot2\,dy = 3\cdot4\,dx - 4\cdot1\,dx$$
$$13\,dy = 8\,dx$$
$$\frac{dy}{dx} = 8/13$$

CALCULATORS ALLOWED—SHOW ALL WORK ON THIS TEST

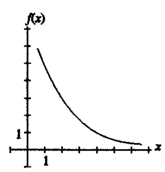
13. Which of the following is an equation of the line tangent to the curve

$$y = \frac{3x}{(x+1)^2}$$

at the point (0,0)?

a) $y = 13x$ b) $y = -2x$ c) $y = -7x$ d) $y = 3x$ e) $y = 8x$

$y' = \frac{1}{(x+1)^4}\left[(x+1)^2(3) - (3x)2(x+1)\right] = \frac{1}{1^4}\left[1^2(3) - (0)(2)(1)\right] = 3$



14. For the graph of the function shown above, which of the following is true?

a) $f' > 0$, $f'' > 0$ b) $f' > 0$, $f'' < 0$ c) $f' < 0$, $f'' > 0$
d) $f' < 0$, $f'' < 0$ e) None of these

15. Suppose $g''(3) > 0$. Which of the following *must* be true?
I. The graph of g is concave up where $x = 3$.
II. g has a local maximum at $x = 3$.
III. g has a local minimum at $x = 3$.
IV. g' is increasing at $x = 3$.
a) I, II, and IV b) I, III, and IV c) I and IV d) I and III e) I only

16. Suppose $g''(3) > 0$. Which of the following *must* be false?
I. The graph of g is concave up where $x = 3$.
II. g has a local maximum at $x = 3$.
III. g has a local minimum at $x = 3$.
IV. g' is increasing at ~~$x = 3$.~~
a) I only b) II only c) III only d) IV only e) None of these

17. The function $h(x) = x^2 - x - 7$ on the interval $[0, 3]$ has a local minimum at which x-~~value?~~
a) 0 b) 1 c) 2 d) 3 e) None of these.

$h'(x) = 2x - 1 = 0$ $h'(3) = 5$ local max

$x = 1/2$

$h'(0) = -1$ local max

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18. The function $f(x) = x^2 - x - 7$ on the interval $[0, 3]$ has an absolute maximum at which x-value?
a) 0 b) 1 c) 2 d) 3 e) None of these.

$f(0) = -7$ $f(1/2) = -\frac{29}{4}$ $f(3) = -1$ ✯

19. If $3 + 4x + 5x^2$ is the second-order Taylor polynomial for the function $g(x)$ centered at $x_0 = 0$, then $g''(0)$ equals:
a) 2 b) 3 c) 4 d) 5 e) 10

20. Which of the following are indeterminate forms?

I. $\frac{0}{0}$ II. $\frac{1}{\infty}$ III. 0^∞ IV. 1^∞

a) I and III b) I and IV c) I, II, and IV d) I, III, and IV e) All of them.

PART TWO: Longer answers (50%). Work the following ten problems. They are not multiple choice. Show all work and your answers on this test paper.

21. a) Find an equation of the straight line containing the points $(4, 1)$ and $(-3, 5)$.

$m = \frac{5-1}{-3-4} = \frac{-4}{7}$

$\boxed{y - 1 = -\frac{4}{7}(x - 4)}$

$y = -\frac{4}{7}x + \frac{16}{7} + \frac{7}{7} = -\frac{4}{7}x + \frac{23}{7}$

b) Find the center and radius of the circle with equation $x^2 + 6x + y^2 - 5y = 0$.

$x^2 + 6x + 9 + y^2 - 5y + \frac{25}{4} = 9 + \frac{25}{4} = \frac{61}{4}$

$(x+3)^2 + (y - 5/2)^2 = 61/4$

$\boxed{\text{center } (-3, 5/2) \quad \text{radius } \frac{\sqrt{61}}{2}}$ $= 3.90512$

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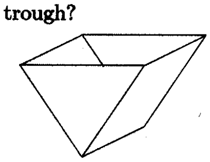
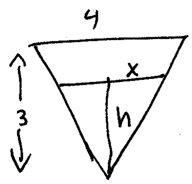
22. A water trough is made such that each cross-section is a triangle (see picture). The depth of the trough is 3 feet and the width at the top is 4 feet. The trough is 10 feet long. Water is poured into the trough at a rate of 1 ft^3 per minute. How fast is the water level rising when the water is 1 foot from the top of the trough?





$\frac{dV}{dt} = 1$ $\boxed{h=2}$

$V = xh(10)$

$\frac{h}{x} = \frac{3}{2} \rightarrow x = \frac{2}{3}h$

$V = \frac{20}{3}h^2$

$\frac{dV}{dt} = \frac{40}{3}h\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{40}\frac{3}{h}\frac{dV}{dt} = \frac{3}{80}\frac{dV}{dt} = \boxed{\frac{3}{80}\ \text{ft/min}}$

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23. Cesium-131 has a half-life of 9.69 days. Initially, a sample of cesium is found to contain 200 grams of cesium-131.

a) Find a formula for the amount of cesium-131 remaining t days after the initial time.

$\boxed{A = 200(2^{-t/9.69})}$ $1/9.69 \approx .1032$ $\frac{\ln 2}{9.69} = .07153$

b) Find an inverse for the above formula and explain what it means.

$\frac{A}{200} = 2^{-t/9.69}$

$\ln\left(\frac{A}{200}\right) = -\frac{t}{9.69}\ln 2$

$t = -9.69\frac{\ln(A/200)}{\ln 2}$
time at which amount reaches A.

c) How much cesium-131 will remain 6 days after the initial time?

$200(2^{-6/9.69}) = \boxed{130.20689}$

d) How many days after the initial time will the amount of cesium-131 be 5 grams?

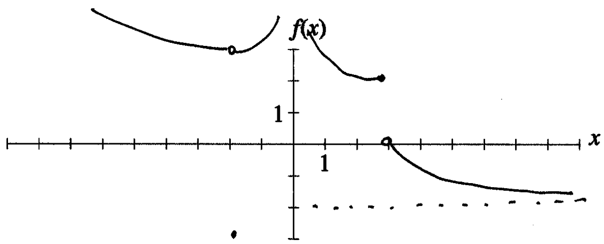
$t = -9.69\frac{\ln(5/200)}{\ln 2} = \boxed{51.569\ \text{days}}$

CALCULATORS ALLOWED—SHOW ALL WORK ON THIS TEST

24. Sketch the graph of a single function satisfying all the following conditions:

$$\lim_{x\to -2} f(x) = 3, \quad f(-2) = -3, \quad \lim_{x\to 3+} f(x) = 0, \quad \lim_{x\to 3-} f(x) = 2,$$

$$\lim_{x\to 0} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = -2, \quad \lim_{x\to -\infty} f(x) = \infty.$$

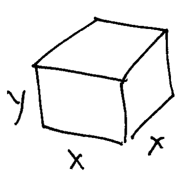


25. A cube made of an unknown substance is known to have a mass of 300 grams. The edge of the cube is measured to be 10 cm with a possible error of ±.03 cm. Use differentials to estimate the corresponding error in the calculated density of the cube.

$$\rho = \frac{m}{e^3} = \frac{300}{e^3}$$

$$d\rho = -\frac{900}{e^4}\, de = -\frac{900}{10^4}(\pm .03) = \boxed{\pm .0027 \frac{g}{cm^3}}$$

26. A closed rectangular box with a square base and a volume of 12 ft^3 is to be constructed using two different types of material. The top is made of a metal costing \$2 per square foot and the remainder is made of wood costing \$1 per square foot. Find the dimensions of the box which minimize the total cost.



$$x^2 y = 12$$

$$2x^2 + x^2 + 4xy = C$$

$$C = 3x^2 + 4xy$$

$$y = \frac{12}{x^2}$$

$$C = 3x^2 + 4x\left(\frac{12}{x^2}\right) = 3x^2 + \frac{48}{x}$$

$$C' = 6x - \frac{48}{x^2} = 0$$

$$x^3 = 8 \quad \boxed{x = 2 \quad y = 3}$$

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27. For the function $f(x) = 5x^6 - 6x^5 - 15x^4 + 10$:

a) Find the intervals on which f is increasing or decreasing on $[-3, 3]$.

$f'(x) = 30x^5 - 30x^4 - 60x^3 = 0$ $\quad$ $f'(x) = 0$ at $x = -1, 0, 2$

$30x^3(x^2 - x - 2) = 0$

$30x^3(x-2)(x+1) = 0$

$\boxed{\begin{array}{ccccccccc} & \text{dec} & & \text{inc} & & \text{dec} & & \text{inc} & \\ -3 & - & -1 & + & 0 & - & 2 & + & 3 \end{array}}$

b) Identify the local maxima and local minima on $[-3, 3]$.

local maxima at $\boxed{\begin{array}{c}(-3, 3898)\\(0, 10)\\(3, 982)\end{array}}$ $\quad$ local minima at $\boxed{\begin{array}{c}(-1, 6)\\(2, -102)\end{array}}$

c) What are the absolute maximum and absolute minimum values taken by the function on $[-3, 3]$?

$\boxed{\text{max } 3898 \quad \text{min } -102}$

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28. Let $f(x) = \ln(x + 1)$.

a) Find the third-order Taylor polynomial T_3 for f centered at $x_0 = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	C_n
0	$\ln(x+1)$	0	$0/0! = 0$
1	$\frac{1}{x+1}$	1	$1/1! = 1$
2	$\frac{-1}{(x+1)^2}$	-1	$-1/2! = -1/2$
3	$\frac{2}{(x+1)^3}$	2	$2/3! = 1/3$

$\boxed{x - \frac{1}{2}x^2 + \frac{1}{3}x^3}$

b) Use this to approximate the value of $\ln(0.8)$.

$\ln(1 - .2) \doteq -.2 - \frac{1}{2}(-.2)^2 + \frac{1}{3}(-.2)^3$

$= \boxed{-.222667}$

CALCULATORS ALLOWED—SHOW ALL WORK ON THIS TEST

29. a) State the limit definition of derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

b) Use the limit definition of derivative to find $f'(x)$ if $f(x) = 2x^2 - 7x - 2$.

$$f'(x) = \lim_{h\to 0} \frac{[2(x+h)^2 - 7(x+h) - 2] - [2x^2 - 7x - 2]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{7x} - 7h - \cancel{2} - \cancel{2x^2} + \cancel{7x} + \cancel{2}}{h} = \lim_{h\to 0} (4x + 2h - 7)$$

$$= 4x - 7$$

30. Let $f(x) = \cos x - x^3$. Use three steps of Newton's method to estimate a value c such that $f(c) = 0$. Start with the initial guess $x_0 = 1$. Use at least 5 decimal places of precision in your calculations.

$f(x) = \cos x - x^3$ $\quad$ $f'(x) = -\sin x - 3x^2$

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	1	−.45970	−3.84147	0.88033
1	0.88033	−.04535	−3.09591	0.86568
2	0.86568	−.0006323	−3.00977	$\boxed{0.86547}$