12 Week Exam
SM121, Section 2043
Calculus 1

29 September 2008

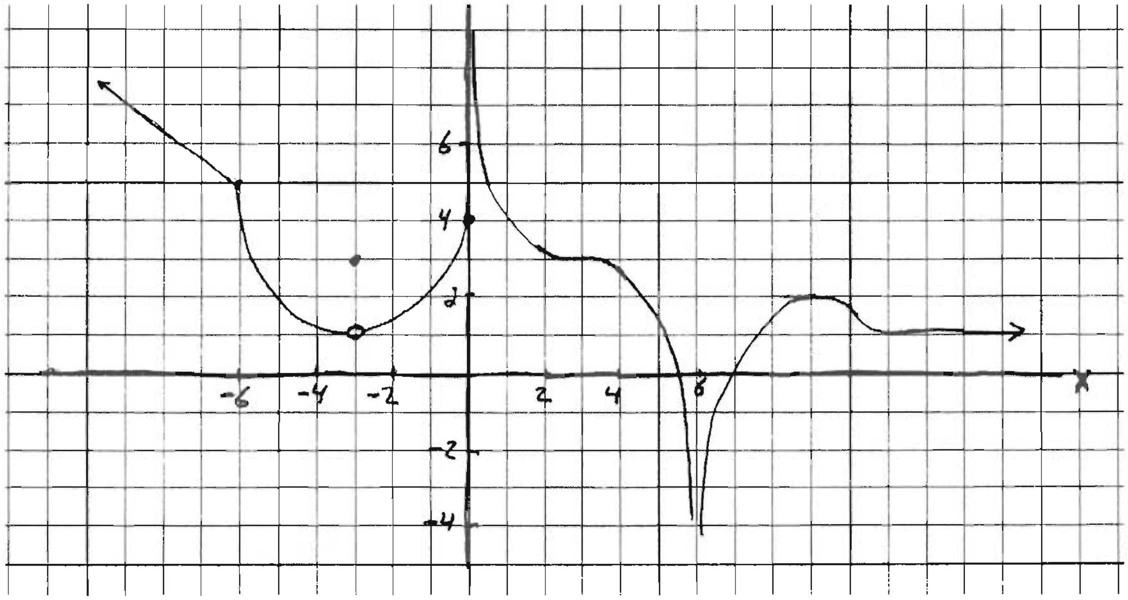
Instructions

**** NO CALCULATORS ****

1. Do not open the exam until instructed.
2. The exam will commence immediately at the beginning of class, or when the entire class is present, whichever occurs first.
3. You may not use any notes, books, calculators, or other materials.
4. You must use pencil only.
5. You may not discuss the exam with anyone, using any medium, until I explicitly give you permission to do so!
6. ***Full credit will only be granted for showing a logical progression of all steps leading to the correct answer.***
7. Graphs must be properly labeled.
8. Simple arithmetic must be computed for full credit. Answers should be clear and complete without further substitutions or re-arrangement.
9. Draw a $\boxed{\text{box}}$ around your final answer.
10. Turn in all scratch paper with your name on each sheet.

Name: Answer Key

1. (15 pts) The graph of $y = f(x)$ is given below.



Evaluate the following:

a) $\lim_{x\to 0^+} f(x) =$ ∞

b) $\lim_{x\to 3} f(x) =$ 3

c) $\lim_{x\to 0^-} f(x) =$ 4

d) $\lim_{x\to\infty} f(x) =$ 1

e) $\lim_{x\to -\infty} f(x) =$ ∞

f) $f'(-8) =$ $-2/3$

g) $f'(9) =$ 0

h) $\lim_{x\to -3} f(x) =$ 1

i) $f(-3) =$ 3

j) For what value(s) of x is f discontinuous?

$x = -3, 0, 6$

k) For what value(s) of x, is f not differentiable?

$x = -6, -3, 0, 6$

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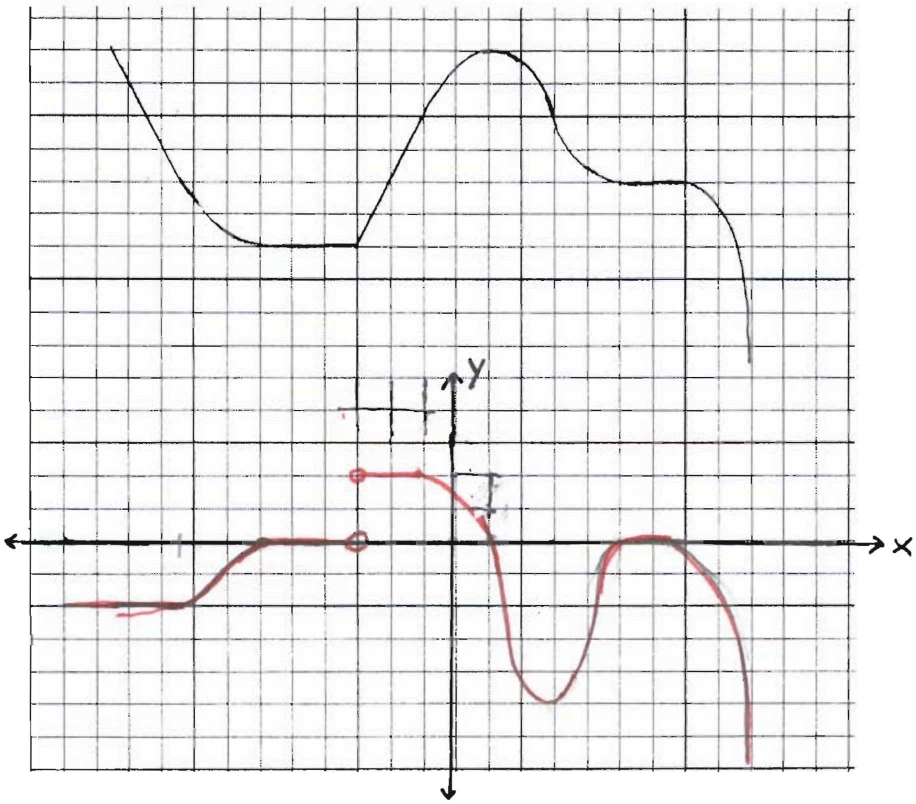
2. (10 pts) Given the graph of $f(x)$ below, sketch the corresponding graph of its derivative, $f'(x)$.



3. (12 pts) Formal definitions:

a) Given function f, state the definition of continuity of f at $x = a$.

$$\lim_{x \to a} f(x) = f(a)$$

b) Given a function f, state the formula for determining the derivative function, f'.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

c) Given function f, state the formula for determining the average rate of change of f over the interval [a, b].

$$\frac{f(b) - f(a)}{b-a}$$

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4. (12 pts) Assuming that gas mileage for a car (*mpg*) is a function of the speed the car is traveling (s), then $mpg = f(s)$.

a) What does $f(55) = 28$ mean? Express your answer with the proper units.

The car's gas mileage is 28 mpg at 55 mph.

b) What does $f'(55) = -\frac{1}{2}$ mean? Express your answer with the proper units.

- Rate of change of gas mileage at 55 mph is $-1/2$ miles per gallon.
- Gas mileage decreases $1/2$ mile per gallon for an increase of 1 mph above 55, or increases $1/2$ mile per gallon for a decrease of 1 mph below 55.

c) Use (a) and (b) above to estimate $f(60)$. Express your answer with the proper units.

$(60-55)(-1/2) = -2.5$ mpg.

$\therefore f(60) \approx 28 - 2.5$

$\boxed{f(60) \approx 25.5 \text{ mpg}}$

5. (8 pts) The number locations of a restaurant chain is given in the table below.

Year	1999	2000	2001	2002	2003
# of restaurants	1,880	2,130	3,400	4,710	5,880

a) Find the average rate of change in the number of restaurants from 1999 to 2003. Express your answer with the proper units.

$$\frac{R(2003) - R(1999)}{2003-1999} = \frac{5880-1880}{4} = \frac{4000}{4} = \boxed{1000 \text{ restaurants/yr}}$$

b) Estimate the instantaneous rate of change in 2001?

$$\frac{R(2001) - R(2000)}{2001-2000} = \frac{3400-2130}{1} = 1270$$

$$\frac{R(2002) - R(2001)}{2002-2001} = \frac{4710-3400}{1} = 1310$$

$$R'(2001) \approx \frac{1270+1310}{2}$$

$\boxed{R'(2001) \approx 1290 \text{ restaurants/year}}$

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6. (9 pts) Find the vertical and horizontal asymptotes of $f(x) = \dfrac{\sqrt{3x^2-2}}{4-7x}$.

Vertical: $4-7x=0$

$$\boxed{x = \frac{4}{7}}$$

Horizontal: $\lim\limits_{x\to\infty} \dfrac{\sqrt{3x^2-2}}{4-7x}$ and $\lim\limits_{x\to-\infty} \dfrac{\sqrt{3x^2-2}}{4-7x}$

$$\lim_{x\to\infty} \frac{\sqrt{3x^2-2}}{4-7x} = \lim_{x\to\infty} \frac{\frac{\sqrt{3x^2-2}}{\sqrt{x^2}}}{\frac{4-7x}{x}}$$ since $x=\sqrt{x^2}$ when $x>0$

$$= \frac{\sqrt{\frac{3x^2-2}{x^2}}}{\frac{4-7x}{x}} = \lim_{x\to\infty} \frac{\sqrt{3-\frac{2}{x^2}}}{\frac{4}{x}-7} = \boxed{-\frac{\sqrt{3}}{7}}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{3x^2-2}}{4-7x}$$

$$= \lim_{x\to-\infty} \frac{\frac{\sqrt{3x^2-2}}{-\sqrt{x^2}}}{\frac{4-7x}{x}}$$ since $x=-\sqrt{x^2}$ when $x<0$

$$= \lim_{x\to-\infty} \frac{-\sqrt{3-\frac{2}{x^2}}}{\frac{4}{x}-7}$$

$$= \frac{-\sqrt{3}}{-7} = \boxed{\frac{\sqrt{3}}{7}}$$

$\therefore$ $\boxed{\text{Horizontal at } y = \pm\frac{\sqrt{3}}{7}}$

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7. (6 pts) Evaluate $\lim_{x\to\infty} \frac{\sin x}{x}$ using the squeeze (sandwich) theorem.

$$-1 \leq \sin x \leq 1$$

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$\lim_{x\to\infty} -\frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\therefore \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

8. (6 pts) Use the Intermediate Value Theorem to show that there is a root of the equation $f(x) = \sqrt[3]{x} - 1 + x$ in the interval (0, 1).

$$f(0) = \sqrt[3]{0} - 1 + 0$$
$$= -1$$

$$f(1) = \sqrt[3]{1} - 1 + 1$$
$$= 1$$

Since $f(0) < 0$ and $f(1) > 0$, there must be a value c, where $0 \leq c \leq 1$, such that $f(c) = 0$.

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9. (8 pts) Given $f(x)$ as defined below, <u>prove</u> whether or not f is continuous or discontinuous at $x = \frac{\pi}{6}$?

$$f(x) = \begin{cases} \sin x & \text{if} \quad x < \frac{\pi}{6} \\ \frac{-15}{\pi}x + 3 & \text{if} \quad x > \frac{\pi}{6} \\ \frac{12}{\pi}x - \frac{3}{2} & \text{if} \quad x = \frac{\pi}{6} \end{cases}$$

f is continuous at $x = \pi/6$ if $\lim_{x \to \pi/6} f(x) = f(\pi/6)$, if they are not equal, then f is discontinuous at $\pi/6$.

$$\left.\begin{aligned} \lim_{x \to \pi/6^-} f(x) &= \lim_{x \to \pi/6^-} \sin x = 1/2 \\ \lim_{x \to \pi/6^+} f(x) &= \lim_{x \to \pi/6^+} \frac{-15}{\pi}x + 3 = \frac{-15}{\pi}\left(\frac{\pi}{6}\right) + 3 = 1/2 \end{aligned}\right\} \lim_{x \to \pi/6} f(x) = \frac{1}{2}$$

$$f(\pi/6) = \frac{12}{\pi}\left(\frac{\pi}{6}\right) - 3/2 = \frac{12}{6} - 3/2 = \frac{1}{2}$$

$$\lim_{x \to \pi/6} f(x) = \frac{1}{2} = f(\pi/6) \quad \therefore \quad f \text{ is continuous at } x = \pi/6$$

10. (6 pts) Compute the derivative of $f(x) = 4x^2 + 2x - 5$ using the derivative formula.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{4(x+h)^2 + 2(x+h) - 5 - (4x^2 + 2x - 5)}{h}$$

$$= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 + 2x + 2h - 5 - 4x^2 - 2x + 5}{h}$$

$$= \lim_{h \to 0} \frac{8xh + 4h^2 - 2h}{h}$$

$$= \lim_{h \to 0} 8x - 2$$

$$\boxed{f'(x) = 8x - 2}$$

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11. (12 pts) Evaluate the following limits. You must show your work!

a) $\lim_{x\to 4} 7x-5$ $= 7(4)-5$

$= \boxed{23}$

b) $\lim_{x\to\infty} \frac{\sqrt{x^3}+3}{x^2+3x+4}$ $= \lim_{x\to\infty} \frac{x^{3/2}+3}{x^2+3x+4} = \frac{\frac{x^{3/2}}{x^2}+\frac{3}{x^2}}{\frac{x^2}{x^2}+\frac{3x}{x^2}+\frac{4}{x^2}}$

$= \lim_{x\to\infty} \frac{\frac{1}{x^{1/2}}+3/x^2}{1+\frac{3}{x}+\frac{4}{x^2}}$

$= \frac{0}{1} = \boxed{0}$

c) $\lim_{x\to -2} \frac{x^2+7x+10}{x+2}$ $= \lim_{x\to -2} \frac{(x+2)(x+5)}{(x+2)}$

$= \lim_{x\to -2} (x+5)$

$= \boxed{3}$

d) $\lim_{x\to -\infty} \frac{(2x+4)(x-3)}{3x^2-5x+2}$ $= \lim_{x\to -\infty} \frac{2x^2-2x-12}{3x^2-5x+2}$

$= \lim_{x\to -\infty} \frac{\frac{2x^2}{x^2}-\frac{2x}{x^2}-\frac{12}{x^2}}{\frac{3x^2}{x^2}-\frac{5x}{x^2}+2/x^2} = \lim_{x\to -\infty} \frac{2-\frac{2}{x}-\frac{12}{x^2}}{3-\frac{5}{x}+\frac{2}{x^2}}$

$= \boxed{\frac{2}{3}}$

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