Set Theory

The mathematical way of talking about collections of things, especially really really big collections.

Basics

1. A set is an unordered collection of objects. Objects can be real objects, like apples or coins, or abstract things like numbers, or even other sets.

2. the book refers to a paradox that arises from this definition. I'll mention it again later, but to avoid that paradox, we'll say that a set cannot be a member of itself.

3. The things in a set are called elements or members.

4. The set of the top three albums on the Billboard Album chart for the week through March 6, 2004 for is:
   {Nora Jones, Kayne West, Evanescence}

5. Some common sets of numbers:

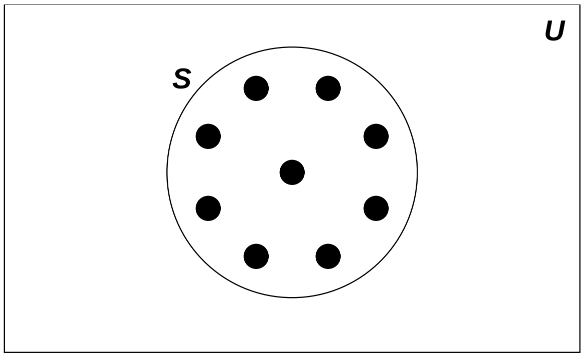
   - $\mathbb{N}$- The natural numbers
   - $\mathbb{Z}$- Integers
   - $\mathbb{Z}^+$- Positive integers
   - $\mathbb{Q}$- Rationals
   - $\mathbb{R}$- Reals

6. Order deosn't matter, and duplicates don't count. The set of primes less than 10 is $\{1, 2, 3, 5, 7\}$. This set is the same as $\{7, 7, 1, 5, 3, 2, 5, 1, 3\}$.

7. two sets are equal if they have the same elements.

8. There are three ways to describe the contents of a set. The first is to list all the elements as we've been doing. The second is "set builder notation," where we describe the elements of the set in some way. This is needed for big or infinite sets. It looks like this:
   $S = \{x \mid x \text{is a prime number}\}$

9. The third way is with a pretty picture, called a Venn diagram, after John Venn. The way a Venn diagram works is we draw a big rectangle, representing the universe of stuff that could be in our set. Then we draw circles to represent our sets, where the elements of the set are inside the circle. Doing this for one set is not very interesting, but it will be useful when we talk about the relations between multiple sets.



10. We define a subset of a set to be a set made up of some of the elements of the set. If A and B are sets, and x is an element
$\forall A, B A \subseteq B \leftrightarrow \forall x, x \in A \rightarrow x \in B$

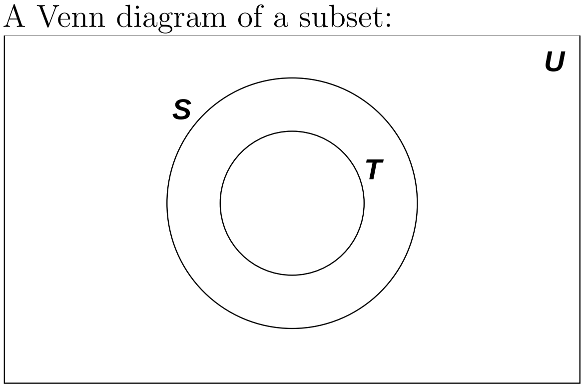
11. The set with no elements is the empty set: $\varnothing$. This is NOT the set containing the empty set.

12. $\varnothing \subseteq S$ and $S \subseteq S$. We can prove these.

13. A proper subset is a subset that is not equal to the other set.
$\forall A, B, A \subset B \leftrightarrow (A \subseteq B) \wedge (A \neq B)$

14. If S is a finite set, then $|S|$ is the cardinality of S, the number of elements of S. If $S = \{a, b, \{c, d\}\}$, then $|S| = 3$.

15. A Venn diagram of a subset:



16. The power set of a set is the set of all subsets. $S = \{a, b, c\}$ then
$P(S) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ if $|S| = n$, then $|P(S)| = 2^n$.

17. An ordered pair is a pair of objects in a particular order: (a, b). An ordered n-tuple has n elements in an order: $(a_1, a_2, a_3, ..., a_n)$

18. The cartesian product of two sets is the set of ordered pairs made from the elements of the two sets, in the order multipled: $A \times B = \{(a, b) \mid a \in A \wedge b \in\}$

19. Give an example of why $A \times B \neq B \times A$

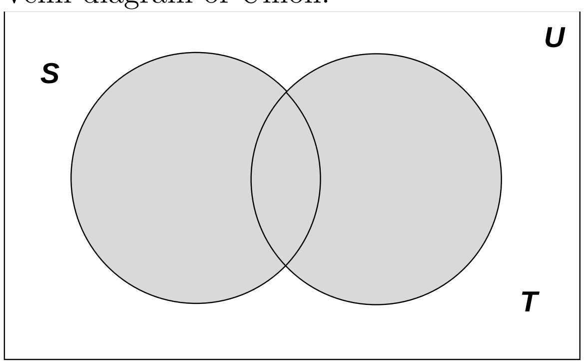
20. $A_1 \times A_2 \times A_3 \times ... \times A_n = \{(a_1, a_2, a_3, ..., a_n) \mid a_i \in A_i\}$

21. lastly, sometimes we'll want to talk about a set that is a subset of another set we know by name like this:
$\forall x \in R \ \ (5 < x < 10)$.

Set Operations

1. The union of 2 sets $A \cup B$ is the set of elements that are in A or B:
$A \cup B = \{x \mid x \in A \vee x \in b\}$

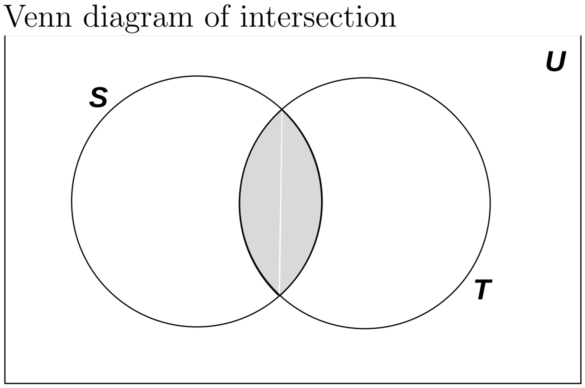
2. Venn diagram of Union:



3. What is the union of $\{1, 2, 3\}$ and $\{2, 3, 4\}$?

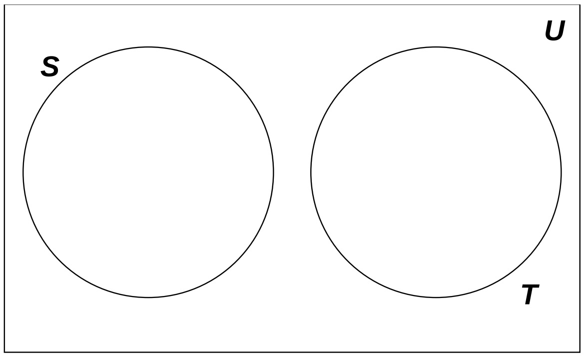
4. The intersection of 2 sets $A \cap B$ is the set of elements in both A and B: $A \cap B = \{x \mid x \in A \wedge x \in b\}$

5. Venn diagram of intersection

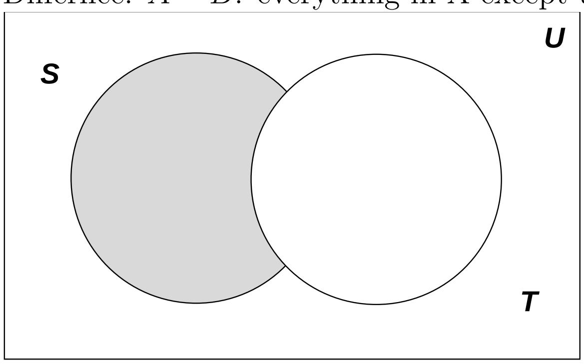


6. What is the intersection of $\{1, 2, 3\}$ and $\{2, 3, 4\}$?

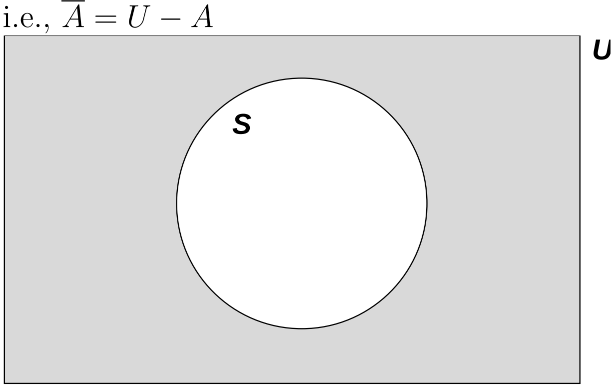
7. Disjoint: $A \cap B = \varnothing$



8. $|A \cup B| = |A| + |B| - |A \cap B|$
9. Differnce. $A - B$. everything in A except those things also in B.



10. Complement. The complement of a set is everything in a set that is not in the universe, i.e., $\overline{A} = U - A$



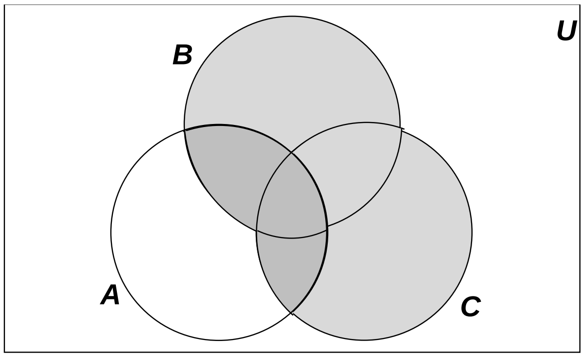
11. Set identities.

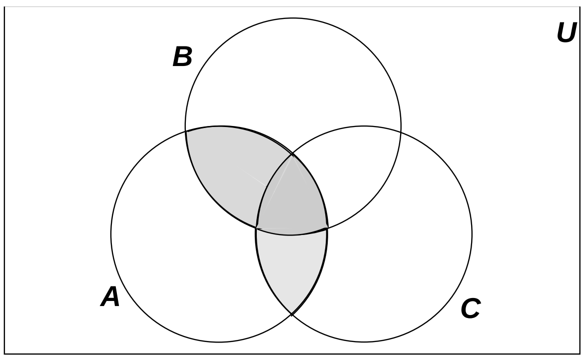
$A \cup \varnothing = A$	Identity
$A \cap U = A$	Identity
$A \cup U = U$	Domination
$A \cap \varnothing = \varnothing$	Domination
$A \cup A = A$	Idempotent
$A \cap A = A$	Idempotent
$\overline{(\overline{A})} = A$	Complementation
$A \cup B = B \cup A$	Commutative
$A \cap B = B \cap A$	Commutative
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative
$A \cap (B \cap C) = (A \cap B) \cap C$	Associative
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive
$\overline{A \cup B} = \overline{A} \cap \overline{B}$	DeMorgan's
$\overline{A \cap B} = \overline{A} \cup \overline{B}$	DeMorgan's
$A \cup (A \cap B) = A$	Absorption
$A \cap (A \cup B) = A$	Absorption
$A \cup \overline{A} = U$	Complement
$A \cap \overline{A} = \varnothing$	Complement

12. Prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$:

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \in \overline{A \cap B}\} \\
&= \{x \mid x \notin A \cap B\} \qquad \text{(definition)} \\
&= \{x \mid \neg((x \in A) \wedge (x \in B))\} \qquad \text{(definition)} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \qquad \text{(DeMorgan's)} \\
&= \{x \mid (x \notin A) \vee (x \notin B)\} \qquad \text{(definition)} \\
&= \{x \mid (x \in \overline{A}) \vee (x \in \overline{B})\} \qquad \text{(Complement)} \\
&= \{x \mid x \in (\overline{A} \cup \overline{B})\} \qquad \text{(definition)} \\
\overline{A \cap B} &= \overline{A} \cup \overline{B}
\end{aligned}$$

13. Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
Lets first convince ouselves its true using Venn diagrams:





$$
\begin{aligned}
A \cap (B \cup C) &= \{x \mid x \in A \cap (B \cup C)\} \\
&= \{x \mid (x \in A) \wedge (x \in (B \cup C))\} \\
&= \{x \mid (x \in A) \wedge ((x \in B) \vee (x \in C))\} \\
&= \{x \mid \ [(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)]\} \\
&= \{x \mid (x \in (A \cap B) \vee (x \in (A \cap C))\} \\
&= \{x \mid x \in (A \cap B) \cup (A \cap C)\}
\end{aligned}
$$

14. Generalized union

$$\bigcup_{i=1}^{n} = A_1 \cup A_2 \cup A_3 \cup ... \cup A_n$$

15. Generalized intersection

$$\bigcap_{i=1}^{n} = A_1 \cap A_2 \cap A_3 \cap ... \cap A_n$$

16. Bit string sets. The most general way to represent sets is as some collection data structure. This has the problem that any data structure in a computer has inherent order, so even testing set equality becomes complicated. On nice solution for finite sets is the bit string representation.

    (a) If we know the universe we're talking about, then we can give the universe some order. For example, if we're talking about the letters of the alphabet, we could use alphabetical order.

(b) We represnt a set in the universe as a string of bits with length equal to the cardinality of the universe. The nth bit indicates whether or not the nth element of the universe is in the set.

(c) For example, if the universe is the letters of the alphabet, and our set is $\{b, e, i, n, o, q, t, x\}$, then the bitstring would be 01001000100001101001000100.

(d) The advantage to this is that set operations are real fast. They can all be done with binary operations: union is &, intersection is |, complement is ˜.

(e) Each of these operations only takes n steps, where n is the number of items in the universe.

Functions

1. A function is an assignment or mapping from the elements of one set to another set. We must assign exactly one element from the second set to each element from the first set. If f is a function from A to B, we write $f : A \to B$.

   (a) One easy example is the assignment of grades to students. One set is students, the other set is grades. For each student, we draw an arrow from that student to a grade.

   (b) Each student has just one arrow coming out, but grades can have multiple arrows coming in. Each student must have a grade.

   (c) Another example is $f : \mathbb{Z} \to \mathbb{Z}$ which maps an integer to the square of that integer: $f(x) = x^2$.

2. A is the *domain* of f.

3. B is the *codomain* of f.

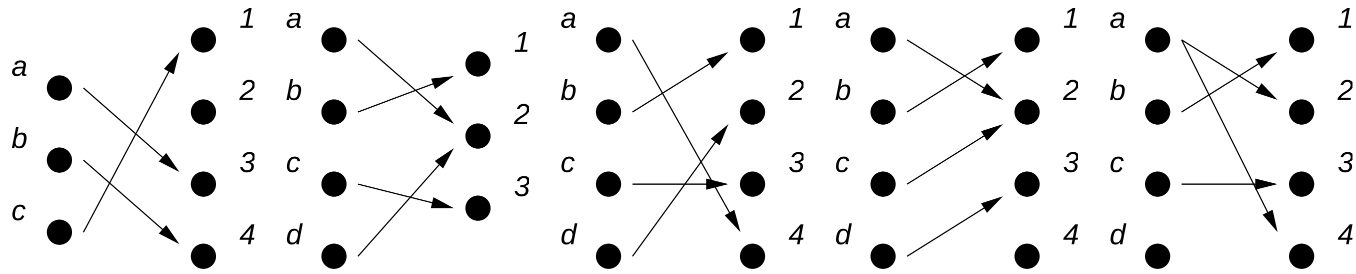
4. for $a \in A$ and $b \in B$ where $f(a) = b$, b is the *image* of a, and a is the *pre-image* of b. The *range* of f is the set of elements in B that are mapped to from A, that is, the set of images from A. f maps A to B.

5. $f(S) = \{f(s) \mid s \in S\}$. That is f of a set S is the set you get passing each element of S through f. S must be a subset of f.

6. Function types:

   (a) One-to-one (injective). A function from A to B where no element is B us used twice. This is written formally as: $f(x) = f(y) \leftrightarrow x = y$. $f(x) = x^2$ is not one-to-one.

   (b) On-to (surjective). A function from A to B where no element of B is ignored. This is written formally as: $\forall b \in B, \exists a \in A, f(a) = b$.



one-to-one onto Both Neither Not a function

1. The composition of two functions: $f(g(x))$ written $(f \circ g)(x)$. If the range of g is not a subset of f, then you can't compose $(f \circ g)$.

2. The inverse of a function is a function with the arrows reversed: $f^{-1}(f(a)) = a$ or $(f^{-1} \circ f)(a) = a$

3. Is $f(x) = x^2$ invertible? Why?