Logic, Propositional

The point of philosophy is to start with something so simple as not to seem worth stating, and to end with something so paradoxical that no one will believe it. -Bertrand Russell

Strictly speaking, logic is rarely used directly in computer science: just a little in Theory and in Artificial Intelligence. But, we will see it is a mathematical model for all sorts of things that go on in CS, including the boolean logic learned in architecture.

Maybe more importantly, logic is the language of proofs. Proofs are important for 2 reasons:

- It is often necessary to prove properties of our programs, and
- proofs themselves *are* programs. To understand one, is to understand the other.

The first kind of logic we will talk about is propositional logic. Propositional logic is one of the simplest logics because it lacks variables.

- A proposition is a statement. Because logic was developed by philosophers, as well as mathematicians, these statements often take the form of sentences:
  - Annapolis is a port.
  - Jolt Cola contains caffeine.
- Propositions also can be mathematical statements:
  - $5 \times 5 = 25$
  - $\sin^2 60 + \cos^2 60 = 1$
- Propositions can be false:
  - Annapolis is larger than Baltimore
  - $6 \times 9 = 42$
  - $\sin^2 60 = (1 - \cos 120)$
- propositions must either be true or false. Questions are not propositions.
- There can be no variables in propositions. For our purposes, the following are not propositions:

- $x + 4 = 2$
- Some midshiman listens to Kenny G.

- propositions can be true or false. Therefore, just as with boolean logic, we can negate them.
  - We will often use the symbol p to indicate some proposition (note that therefore, p is a variable. Thats OK, since p is not the proposition, p represents a proposition. It is interesting to note that it is impossible to talk *about* propositional logic using propositional logic).
  - to negate a proposition, we will use the $\neg$ symbol as in: $\neg p$.
  - so if p is $6 \times 9 = 42$, the $\neg p$ is $6 \times 9 \neq 42$
  - This we have built a simple expression using a proposition.

- A Truth table is a table that describes the values expressions built out of propositions can take:

p	$\neg p$
T	F
F	T

- We can also connect propositions into expressions using *and*, represented by $\wedge$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$2 \times 3 = 6 \wedge 6 \times 9 = 42$ is?

- Along with and, we also have *or*, represented by $\vee$. You can remember the differenece because the $\wedge$ looks like the A in *and*. The truth table is:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- We also can talk about the exclusive-or $\oplus$, which has a table of:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- Implications
  - Predicate logic goes beyond boolean logic in that it contains implication $\rightarrow$.
  - Implication is written $a \rightarrow b$, and is read "a implies b" or "if a then b", or many others in the text.

- it is a statement that if you know a to be true, then you know b to be true.
- Here is the truth table:

a	b	$a \to b$
T	T	T
T	F	F
F	T	T
F	F	T

- the first 2 lines make the most sense. If a is true and b is true, then $a \to b$ is true. If a is true and b is false, then $a \to b$ is is not a true statement.
- The second 2 lines strike most people as a little wierd. If a is false, how can we say anything about $a \to b$? You would almost what to say, "don't know" as the value. Now, unfortunately, we're working in logic, so we can't say "don't know", we can only say true or false, so we'll have to pick one.
- So which one? Logicians like to think of an implication as a pledge, or promise. The value can only be false if the promise is broken. If the antecedent is false, there is no way the pledge can be broken. "If you get an A, I'll buy you a car." If you actually get a B, then I can either buy you a car or not, either way, I did not break my promise. It turns out that deep in the bowels of logic, there are good reasons for interpreting it like this, but only people with graduate degrees in logic care.
- Implication only goes one way. $a \to b$ does not necessarily mean $b \to a$. Sometimes it is true, so we have a new symbol for that: $\leftrightarrow$. $a \leftrightarrow b \Leftrightarrow (a \to b) \wedge (b \to a)$.

a	b	$a \leftrightarrow b$
T	T	T
T	F	F
F	T	F
F	F	T

- Is the contrapositive equivalent? $a \to b \Leftrightarrow \neg b \to \neg a$?
- How about this? $a \to b \Leftrightarrow \neg a \vee b$?
- True implications are not always causal, like: "If I don't like you, you won't get an A". Sometimes a and b are otherwise related: "If you gte an A, I'll say nice things to your parents". Or sometimes they're just random: "If 2+2=9, then I'm a duck".

• Examples. Give the most specific formulation for:

1. I don't want to go to class.
2. The sky is blue or the sea is green
3. Either you back off, or I'll punch you in the nose.
4. It is 5 a.m. and you are listening to Los Angeles
5. Stickshifts and safetybelts, bucket seats have all got to go.
6. If you don't back off, I'll punch you in the nose.

7. you can do that–if you're wound up.
8. 3 is not less than 4, or $6 < 8$.
9. You will fly away only if you want to.
10. If you don't know the answer, you didn't pay attention or you're not very clever.
11. If you eat at King Hall, you will have a stomach ache, unless you have an iron stomach..

- Applications:
  - Translation of english sentences for logical arguments. http://philarete.home.mindspring.com/philosophy/freewill.html describes a classic example of translating english to logic, and then drawing a conclusion from that.
  - Providing precise specifications. We state all the things we want to be true about our new computer system. Unfortunately we don't know if any contradict each other. We translate them to logic statements and have a computer program test for consistency.
  - Look at the many examples in your text.

- Tautologies, Contradictions, Equivalences.
  - Tautology- compound proposition that is always true, no matter what the values of the base propositions.
  - Two ways to show a tautology: draw the truth table:

b	$\neg b$	$b \vee \neg b$
T	F	T
F	T	T
T	F	T
F	T	T

or prove. We'll talk about proving it in a sec.

  - Contradiction- compound proposition that is always false, no matter what the values of the base propositions.
  - Again, there are two ways to show a contradiction-: draw the truth table:

b	$\neg b$	$b \wedge \neg b$
T	F	F
F	T	F
T	F	F
F	T	F

  - Equivalences- two compound propositions are logically equivalent if $p \leftrightarrow q$ is a tautology. That is, both columns in the table are identical:

a	b	$\neg a$	$a \rightarrow b$	$\neg a \vee b$	$(a \rightarrow b) \leftrightarrow (\neg a \vee b)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

- Once you have equivalences you can use them to generate new equivalences by repeatedly replacing pieces of a compound proposition with equivalent propositions.
- Some common equivalences we can generate using tables:

formula	equivalence	name
$p \wedge T$	p	And-Identity
$p \vee F$	p	Or-Identity
$p \vee T$	T	Or-domination
$p \wedge F$	F	And-domination
$p \vee p$	p	Idempotent
$p \wedge p$	p	Idempotent
$\neg(\neg p)$	p	Double Negation
$p \vee q$	$q \vee p$	Commutative
$p \wedge q$	$q \wedge p$	Commutative
$(p \vee q) \vee r$	$p \vee (q \vee r)$	Associative
$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$	Associative
$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$	Distributive
$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	Distributive
$\neg(p \vee q)$	$\neg p \wedge \neg q$	DeMorgan's
$\neg(p \wedge q)$	$\neg p \vee \neg q$	DeMorgan's
$p \vee (p \wedge q)$	p	Absorption
$p \wedge (p \vee q)$	p	Absorption
$p \vee \neg p$	T	Negation
$p \wedge \neg p$	F	Negation
$p \rightarrow q$	$\neg p \vee q$	I
$p \rightarrow q$	$\neg q \rightarrow \neg p$	Contra-positive
$p \vee q$	$\neg p \rightarrow q$	III
$p \wedge q$	$\neg(p \rightarrow \neg q)$	IV
$\neg(p \rightarrow q)$	$p \wedge \neg q$	V
$(p \rightarrow q) \wedge (p \rightarrow r)$	$p \rightarrow (q \wedge r)$	VI
$(p \rightarrow r) \wedge (q \rightarrow r)$	$(p \vee q) \rightarrow r$	VII
$(p \rightarrow q) \vee (p \rightarrow r)$	$p \rightarrow (q \vee r)$	VIII
$(p \rightarrow r) \vee (q \rightarrow r)$	$(p \wedge q) \rightarrow r$	IX
$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$	X
$p \leftrightarrow q$	$\neg p \leftrightarrow \neg q$	XI
$p \leftrightarrow q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	XII
$\neg(p \leftrightarrow q)$	$p \leftrightarrow \neg q$	XIII
$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$	chain
$(p \oplus q) \wedge q$	$\neg p$	mutual exclusion
$p \oplus q$	$(p \vee q) \wedge \neg(p \wedge q)$	XIV
$p \oplus q$	$\neg(p \leftrightarrow q)$	XV

- To show that a proposition is a tautology, repeatedly replace pieces of the proposition using the equivalences, until the proposition is reduced to a single T.

proposition	reason
$(p \leftrightarrow q) \to (p \wedge q) \vee (\neg p \wedge \neg q)$	Given
$[(p \to q) \wedge (q \to p)] \to (p \wedge q) \vee (\neg p \wedge \neg q)$	X
$[(\neg p \vee q) \wedge (\neg q \vee p)] \to (p \wedge q) \vee (\neg p \wedge \neg q)$	I
$\neg\,[(\neg p \vee q) \wedge (\neg q \vee p)] \vee [(p \wedge q) \vee (\neg p \wedge \neg q)]$	I
$[\neg(\neg p \vee q) \vee \neg(\neg q \vee p)] \vee [(p \wedge q) \vee (\neg p \wedge \neg q)]$	DeMorgan's
$[(p \wedge \neg q) \vee (q \wedge \neg p)] \vee [(p \wedge q) \vee (\neg p \wedge \neg q)]$	DeMorgan's
$(p \wedge \neg q) \vee (q \wedge \neg p) \vee (p \wedge q) \vee (\neg p \wedge \neg q)$	Associative
$(p \wedge \neg q) \vee (p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$	Commutative
$[p \wedge (q \vee \neg q)] \vee [\neg p \wedge (q \vee \neg q)]$	Distributive
$[p \wedge T] \vee [\neg p \wedge T]$	Negation
$p \vee \neg p$	And-Identity
T	Negation
q.e.d.	

- To show that a proposition is a contradiction, repeatedly replace pieces of the proposition using the equivalences, until the proposition is reduced to a single F.
- To show that two propositions are equivalent, repeatedly replace pieces of one of the propositions using the equivalences, until that proposition matches the other.