Logic, Predicate

Propositional can be used for some things, but it's somewhat limited. In particular, we can't use variables. This prevents us from making general claims: "Midshipmen are persons of integrity." while still being able to reason about the pieces. Nor can we make claims about something being true without a specific example: "There is decent pizza in Annapolis." To translate either of these statements into logic, we need variables. When we have variables, we have Predicate Logic.

1. Variables
   (a) In Predicate logic, the sentences have variables. The simplest cases are ones you already know: algebra. $x + 3 = 10$, etc.
   (b) Instead of talking about a or b as propositions, we represent statements as follows: Q(x), where Q is a predicate (A proposition with a variable, aka propositional function), and x is a variable. Usually, predicates are capitalized, and variables are lower case.
   (c) A predicate is true or false, depending on the values the variables take. if Q(x) is $x + 3 = 10$, then Q(x) is only true when x=7.
   (d) If we assign an individual value to a variable, the predicate becomes a proposition.
2. Quantifiers
   (a) There is another way to talk about many possible values of variable. This is Quantification.
   (b) Imagine the statement $P(x) : x < x + 1$. When is this true? always, of course. To say that the statement is true for all values of x, we use a Universal Quantifier: $\forall x P(x)$. "For all $x, P(x)$."
   (c) if $Q(x)$ is $x + 3 = 10$, is $\forall x, Q(x)$ true?
   (d) When we say $\forall x$ in the above statements, we were talking about numbers, right? But if x is a truly universal variable, one we can use to talk about more than math, we should be able assign any value to it. What if $x =$ThisDocument ? is it true that ThisDocument $<$ ThisDocument+1? it's not even meaningful to say that. In the above examples, we implicitly meant "in the realm of numbers". This realm is what we'll call the universe of discourse. The universe is usually set before we start talking about statements, and we'll assume the values of all the variables can only be selected from the universe. Sometimes we (and the textbook) will be a little sloppy with the universe specification, but only in cases where it should be obvious.
   (e) So, if I say the universe is all even numbers, is it true to say that $\forall x, x \mod 2 = 0$?
   (f) When we're trying to find the solution to a problem, such as $x + 3 = 10$ what we're asking is, does there exist a value or values for x that makes $Q(x)$ true, or $\exists x, Q(x)$? In general, $\exists x, Q(x)$ says that there exists an x to satisfy that statement.
   (g) T or F: $\exists x, x > x + 1$.
   (h) We can do 3 things with a variable: assign a single value to it, quantify it Universally, or quantify Existentially. If you do any of these things to it, the variable is *bound*. Otherwise, the variable is *free*. Free variables are "out of scope".
   (i) $\exists x, Q(x, y)$. y is out of scope.
   (j) $\exists x(P(x) \vee Q(x)) \wedge \forall x(R(x))$ The universally quantified x overides the existential inside the scope.
3. Functions

(a) Just like programming functions or mathematic functions, logic functions take an argument and return a value.

(b) They look just like predicates: $F(x)$. Must be clear abot which is which.

(c) We use them to talk about objects with unknown names:

$$HasSister(joe1)$$
$$Tall(EldestSisterOf(joe1))$$

$EldestSisterOf()$ returns the eldest sister of the object that is the argument. But we could have just created a new constant: $Tall(joe1sEldestSister1)$.

(d) But that won't work with quantification. If the universe is midshipmen: $\forall x, Tall(EldestSisterOf(x))$. "All midshipmens' eldest sisters are tall.

(e) reasonable rare.

(f) Universe is people: $\forall x, y, Midshipman(x) \wedge EldestSisterof(y, x) \rightarrow Tall(y)$.

4. Negation

   (a) If our universe is CS majors, then $\forall x, TakesSI262(x)$?

   (b) "Every CS major takes SI262."

   (c) What if we negate this? "It is not true that every CS major takes SI262." $\neg\forall x, TakesSI262(x)$.

   (d) This is the same as, "some CS major does not take SI262." or "There exists some CS major who does not take SI262", or $\exists x, \neg TakesSI262(x)$.

   (e) Therefore, $\neg\forall x, TakesSI262(x) \equiv \exists x, \neg TakesSI262(x)$.

5. English to logic. Our universe is midshipmen.

   (a) Everyone is in the Navy. $\forall x, InNavy(x)$.

   (b) All firsties are smart. $\forall x, Firstie(x) \rightarrow Smart(x)$. Note the use of a predicate to indicate which midshipmen, and the implication. Not, $\forall x, Firstie(x) \wedge Smart(x)$. What does that say? (build table)

   (c) Some are well dressed. $\exists x, WellDressed(x)$.

   (d) Some firsties are arrogant. $\exists x, Firstie(x) \wedge Arrogant(x)$. Not, $\exists x, Firstie(x) \rightarrow Arrogant(x)$. Why is this always true? (build table)

   (e) Every Firstie has done CSNTS or Plebe Summer. $\forall x, Firstie(x) \rightarrow CSNTS(x) \vee PlebeSummer(x)$.

   (f) "A CS major was given Patriot League Swimmer Of The Week honors this week." $\exists x, CSMajor(x) \wedge GivenPatriotLeagueSwimmerOfTheWeek(x)$, or maybe:
   $\exists x, CSMajor(x) \wedge ReceivedAwardAt(x, patriotLeagueSwimmerOfTheWeek, thisWeek)$

6. Nested Quantifiers. What happens when we have multiple variables in a sentence? Insanity.

   (a) We'll start with math, because its easy. (Thats is, our universe will be real numbers).

   (b) When we make property claims, we often use 2 Universal quantifiers: $\forall x \forall y, xy = yx$. The commutative property of multiplication.

   (c) but sometimes not. $\forall x \exists y, x + y = 0$ is the property of additive inverse. This says, for all numbers, there exists a number that can be added to it to make the sum 0.

   (d) $\exists x \forall y, x + y = 0$. This says, there exists some number such that no matter what number I add to it, I get 0. Is that true?

   (e) Order makes a huge difference.

   (f) What's the distributive law?

(g) The type and order of the quantifiers used in describing the laws determine how and when we can use the law. You understand these properties inuitively, without the quantifiers, because you've been working with the laws for 10 years. But for a computer to know when to apply the law, it must be given the quantifiers. This is one reason people bother to attach the quantifiers to these mathematical laws. Without them, computers could not do higher level math!

(h) Appropriate for this class: $\forall x(HasText(x) \vee \exists y(HasText(y) \wedge Friend(x,y))$.

(i) Note that we can move the middle $\exists$ out to the front, BUT NOT PAST THE $\forall$! What would that mean?

(j) x is USNA professors, y, z are midshipmen $\exists x \forall y \forall z(HasInClass(x,y) \wedge HasInClass(x,z) \wedge (y \neq z) \rightarrow SameRank(y,z))$?

(k) What about going the other way? ($\exists$ moved to the right...)

(l) Every prime can be expressed as the sum of two other primes.$\forall x(Prime(x) \rightarrow \exists y \exists z(Prime(y) \wedge Prime(z) \wedge x = y + z))$.

(m) Universe people. Everyone likes some kind of PopTart. $\forall x \exists y(Likes(x,y))$.

(n) There is a single kind of PopTart which everyone likes.

(o) Note that the logic expresses some ideas much more precisely than english.

7. Nesting negated quantifiers. The rules are the same as before: pushing a neagation across a quantifier changes its type.

   (a) There is no midshipman who has failed to turn in all SI262 Homeworks.

$$
\begin{aligned}
&\neg\exists x(Midshipman(x) \wedge \forall y(SI262Homework(y) \rightarrow \neg TurnIn(x,y))) \\
&\neg\exists x \forall y(Midshipman(x) \wedge (SI262Homework(y) \rightarrow \neg TurnIn(x,y))) \\
&\neg\exists x \forall y(Midshipman(x) \wedge (\neg SI262Homework(y) \vee \neg TurnIn(x,y))) \\
&\forall x \neg\forall y(Midshipman(x) \wedge (\neg SI262Homework(y) \vee \neg TurnIn(x,y))) \\
&\forall x \exists y \neg(Midshipman(x) \wedge (\neg SI262Homework(y) \vee \neg TurnIn(x,y))) \\
&\forall x \exists y(\neg Midshipman(x) \vee \neg(\neg SI262Homework(y) \vee \neg TurnIn(x,y))) \\
&\forall x \exists y(\neg Midshipman(x) \vee (SI262Homework(y) \wedge TurnIn(x,y))) \\
&\forall x \exists y(Midshipman(x) \rightarrow (SI262Homework(y) \wedge TurnIn(x,y)))
\end{aligned}
$$

   All midshipmen have turned in at least 1 SI262 homework.

   (b) See? piece of cake ;-)

8. This is a really useful table taken from your text:

Statement	When True	When False
$\forall x \forall y(P(x,y))$ $\forall y \forall x(P(x,y))$	true for every pair of x, y	There is at least one pair of x, y for which $P(x,y)$ is false
$\forall x \exists y(P(x,y))$	For all xs, there is at least 1 y for which $P(x,y)$ is true.	There is some x where P(x,y) is false for *every* y
$\exists x \forall y(P(x,y))$	There is some x for which *all* ys make $P(x,y)$ true.	For every x, there is at least 1 y for which $P(x,y)$ is false.
$\exists x \exists y(P(x,y))$ $\exists y \exists x(P(x,y))$	There is some pair x, y for which $P(x,y)$ is true.	$P(x,y)$ is false for all pairs of x, y.