Permutations and Combinations

1. A permutation is a ordering of elements in a set, such as the ranking of the students in your class. An r-permutation is an ordering of r elements selected from the set.

   (a) The number of r-permutations of a set with n elements is:

   $$P(n,r) = n(n-1)(n-2)...(n-r+1)$$

   There are n ways to pick the first element, $n-1$ ways to pick the second, $n-2$ to pick the third and so on until the rth element, for which there are $(n-r+1)$ to pick. The pruduct rule says we multiply these together, thus getting the above equation. Note that this can be simplified to:

   $$P(n,r) = \frac{n!}{(n-r)!}$$

   (b) How many ways can we permute the letters ABCDEFGH so that ABC occurs next to each other? If ABC is contiguous, then it is a single unit, thus there are really only 6 things to order: ABC D E F G H. Thus the number of permutations is 6! or 720.

   (c) How many ways can we permute the letters ABCDEFGH so that ABC occur in order, but not necessarily next to each other? Think of this NOT as choosing letters to fill slots, but choosing slots for letters. For the letter D, we have 8 slots, for the letter E we have 7 slots, etc. This is a case of P(8,5). Once we're done with all those letters, how many ways can we put ABC in there? There are 3 slots left, and they must be in order, so 1 way. Therefore, there are P(8,5) or 6720 ways.

2. A r-combination is an unordered set of r elements selected from a set of elements.

   (a) The number of r-combinations of a set of n elements is:

   $$C(n,r) = \frac{n!}{r!(n-r)!}$$

why?

$$P(n,r) = C(n,r)P(r,r)$$

The number of ways we can select, times the number of orderings.

$$C(n,r) = \frac{P(n,r)}{P(r,r)}$$

$$C(n,r) = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}}$$

$$C(n,r) = \frac{n!}{(n-r)!} \cdot \frac{1}{r!}$$

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

(b) How many ways can we select 5 students from this class, in no particular order?

(c) 13 people on a softball team show up for a game. How many ways can we choose 10 players? How many ways can we assign the 10 positions? 3 of the 13 are women. Since this is a co-ed league, how many ways can we pick 10 where at least 1 is a woman?

(d) $C(n,r) = C(n, n-r)$.

3. Repetitions

   (a) If we can repeat elements, then the number of r-permutations is n^r.

   (b) The number of alphabetic strings of length n is 26^n.

   (c) the number of r-combinations is C(n+r-1,r).

   (d) The ACM meeting has 5 kinds of pizza. how many ways can 3 pieces be chosen? C(7,3).

   (e) How many solutions are there to the equation $x + y + z = 11$? Think about his in terms of what the equation represents: You have a backpack that can hold eleven items. You want to fill your backpack with water bottles, granola bars, and teddy bears. How many ways can you do this? How many combinations are there to select these 11 items? This is a case of an 11-combination with repeats, or $C(11 + 3 - 1, 11) = C(13, 11) = C(13, 2) = 78$

4. Permutations with indistinguishable objects.

   (a) What about the case where we want a permutation, but some elements are indistinguishable? That is since *some* elements are exactly alike, we could swap those elements and get the same sequence.

   (b) Common examples are anagrams. How many ways can we re-arrage the letters in SUCCESS? 'MICROSOFT TECHNICAL SUPPORT'? It's not permutations, because swapping two identical letters ends up with the same string. Its not combinations because swapping two different letters ends up with a different string. (Con, from culprit's pathetic OS)

(c) Think of the word as a set of slots to which to assign letters. The 3 S's can go any of 7 positions, in any order, or C(7,3). Then the 2 C's can go in any of the remaining 4 positions, or C(4,3). The E can go in any of 2 positions, C(2,1), and the U can go in any of 1 position C(1,1). This is: $C(7,3)C(4,3)C(2,1)C(1,1)$ by the product rule, which reduces to:

$$\frac{n!}{n_1!n_2!...n_k!}$$

where there are n objects, n_1 of type 1, n_2 of type 2,...,n_k of type k.

5. Distributing distinguishable objects into distinguishable boxes.

   (a) The number of ways of distributing n objects into k boxes so that n_i objects are in box i is:

$$\frac{n!}{n_1!n_2!...n_k!}$$