Induction

Beyond the natural numbers, addition, multiplication, and mathematical induction are intuitively clear. - Luitzen Brouwer (1881-1966)

There is a special proof technique, one that you've probably seen before, called induction. This proof technique is intimetely tied together with recursion, which we know and love so much.

1. Induction is used to prove statements of the form $\forall x(P(x))$.
2. The universe for x must have a special property: it must be well ordered.
3. Well ordered is a property defined over set of objects (such as a variable's universe) that says that for every sub-set of that set, there is a minimum. What this means is you can take the set, find the minimum, remove it, and find the minimum of the remaining sub-set, etc. This provides an ordering from minumum to maximum of the set.
4. obviously, the naturual numbers are well ordered.
5. Any set from which we can make a one to one mapping to the natural numbers is therefore also well-ordered.
6. Mathematical induction, relying on this property, is used to show that all members of a well-ordered set have some property. It has the following 2 steps:
   (a) Basis step: Show that the property holds for the minimum of the set, i.e. $P(1)$ is true.
   (b) Induction step: Show that the implication $P(k) \rightarrow P(k+1)$ is true for all k.
   (c) Concisely summarized as: $(P(1) \wedge \forall k(P(k) \rightarrow P(k+1))) \rightarrow \forall n(P(n))$.
   (d) Does this make sense? discuss.
   (e) Hopefully this process reminds you of something.
7. What is the sum of the first n odd positive integers?
   $1 = 1$ $\quad 1+3=4$ $\quad 1+3+5=9$
   $1+3+5+7=16$ $\quad 1+3+5+7+9=25$ $\quad 1+3+5+7+9+11$
   (a) We have good reason to believe the answer isn^2. But haw to prove it?
   (b) By induction! duh.
   (c) Basis step: P(1) is $1^2 = 1$. Is that true? yep.
   (d) Induction step: if $k^2 = 1+3+5+...+(2k-3)+(2k-1)$ then $(k+1)^2 = 1+3+5+...+(2k-1)+(2k+1)$

$$1+3+5+...+(2k-1)+(2k+1) = (k+1)^2 \tag{1}$$
$$[1+3+5+...+(2k-1)]+(2k+1) = (k+1)^2 \tag{2}$$
$$k^2+(2k+1) = (k+1)^2 \tag{3}$$
$$(k^2+2k+1) = (k+1)^2 \tag{4}$$
$$(k+1)(k+1) = (k+1)^2 \tag{5}$$
$$(k+1)^2 = (k+1)^2 \square \tag{6}$$

8. Show that $n < 2^n$.

$$\text{Basis:}$$
$$1 < 2^1 \tag{7}$$
$$\text{Induction:}$$
$$k < 2^k \tag{8}$$
$$2k < 2^k 2 \tag{9}$$
$$2k < 2^{k+1} \tag{10}$$
$$1 \leq k \tag{11}$$
$$k + 1 \leq 2k \tag{12}$$
$$\therefore k + 1 < 2^{k+1} \square \tag{13}$$

9. $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

$$\text{Basis:}$$
$$1 = 1(1 + 1)/2 \tag{14}$$
$$\text{Induction:}$$
$$1 + 2 + 3 + ... + k = \frac{k(k + 1)}{2} \tag{15}$$
$$(1 + 2 + 3 + ... + k) + k + 1 = \frac{(k + 1)((k + 1) + 1)}{2} \tag{16}$$
$$\frac{k(k + 1)}{2} + k + 1 = \frac{(k + 1)(k + 2)}{2} \tag{17}$$
$$\frac{k^2 + k)}{2} + \frac{2k + 2}{2} = \frac{(k + 1)(k + 2)}{2} \tag{18}$$
$$\frac{k^2 + 3k + 2}{2} = \frac{(k + 1)(k + 2)}{2} \tag{19}$$
$$\frac{(k + 1)(k + 2)}{2} = \frac{(k + 1)(k + 2)}{2} \square \tag{20}$$